ALGORITHM FOR EXTRACTION OF GEOMORPHOLOGICAL CHARACTERISTICS BASED ON A MATHEMATICAL FUNCTION SURFACE DERIVED FROM BS-HORIZON

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ABSTRACT

BS-Horizon is one of the surface estimation programs for geologic boundary surfaces and geomorphic surfaces. This program makes it possible to keep the estimated surface as a bi-cubic B-spline function. For the practical use of this surface, we developed a new algorithm to extract geomorphological characteristics. This algorithm, coded in a FORTRAN program, is designed to extract the value of the surface (height), the first and second partial derivatives of the surface and some major geomorphological characteristics based on the information of the surface produced by BS-Horizon. In this paper, we explained the detail of algorithm and demonstrated two examples of extraction. The results revealed that the algorithm is useful for continuous and exact analysis related to the surface, and indicated that the surface produced by BS-Horizon greatly contributes to improvement of the quantitative analyses in the field of geomorphology and geology.

1. INTRODUCTION

Over the past few decades, a considerable number of studies have been conducted on utilization of DEM (Digital Elevation Model) in geomorphologic analysis. In the analysis using DEM, the quantification values of topography based on some definition, called "geomorphological characteristics", are usually used. There are many kinds of geomorphological characteristics that are defined with the aid of the partial derivatives or definite integration of the surface. Because DEM is a set of discrete values that give heights of the surface at grid nodes, numerical differentiation and numerical integration are generally used in the calculation of such values. However such a discrete valuable calculation methods does not avoid some approximation errors. The problems in data and calculation become a great obstacle to quantitative analyses related to the surface.

On the other hand, the surface estimation program BS-Horizon (Nonogaki *et al.*, 2008) has a capability not only to produce a DEM but also to define a surface in a form of bi-cubic B-spline function, which gives a continuous surface up to the partial derivatives of second order. Therefore, we can calculate directly the geomorphological characteristics without any approximation. The purpose of this paper is to present an algorithm for extraction of geomorphological characteristics defined with the aid of the partial derivatives and definite integration based on the information of the surface produced by BS-Horizon using the feature of cubic B-spline.

2. ALGORITHM

2.1 Information of the surface produced by BS-Horizon

First of all, suppose that a surface can be expressed by $z = f(x, y)$ in a Cartesian coordinates with the x-axis pointing towards the east, the y-axis pointing towards the north and the vertical z-axis. Let $\Omega = \Omega_x \times \Omega_y$ (Ω_x: $x_{\min} \le x \le x_{\max}$, Ω_y: $y_{\min} \le y \le y_{\max}$) be rectangular domain in x-y plane. Provided that Ω is divided into $M_x \times M_y$ sections and that the knots s_i ($i = -3, -2, \ldots, M_x + 3$) and t_j ($j = -3, -2, \ldots, M_y + 3$) are equally set as shown in Figure 1, the surface in Ω is given by

$$f(x, y) = \sum_{i=1}^{M_x+3} \sum_{j=1}^{M_y+3} c_{ij} N_i(x) N_j(y) \tag{1}$$

where $N_i(x)$ and $N_j(y)$ are normalized cubic B-spline bases with respect to x and y respectively, and c_{ij} are the coefficients for each tensor product of bases that BS-Horizon determines in surface estimation. Among the information of the surface mentioned above, BS-Horizon produces (1) a coordinate of south-west edge of Ω: ($x_{\min}$, $y_{\min}$), (2) the number of divisions: M_x and M_y, (3) the knot intervals: h_x and h_y given by

$$h_x = (x_{\max} - x_{\min}) / M_x, \; h_y = (y_{\max} - y_{\min}) / M_y \tag{2}$$

where ($x_{\max}$, $y_{\max}$) is a coordinate of north-east edge of Ω, and (4) the coefficients of bases: c_{ij}.

2.2 Relation between cubic B-splines and cubic polynomials in non-zero sections

In order to understand the bi-cubic B-spline function, we describe several natures of the cubic B-spline bases with respect to x. The same applies to the one with respect to y.

A cubic B-spline has non-zero value only in four continuous sections. For example, the basis $N_{i+3}(x)$ becomes non-zero only in section i to section $i + 3$ ($s_{i-1} \le x \le s_{i+3}$) as shown in Figure 2. This means that only four bases $N_i(x)$, $N_{i+1}(x)$, $N_{i+2}(x)$ and $N_{i+3}(x)$ have non-zero value in section i ($s_{i-1} \le x \le s_i$) as shown in Figure 3. On the other hand, let x be a point which exists in section i ($s_{i-1} \le x \le s_i$) (See Figure 3), and let r_x be a coordinate of x measured from left neighbor knot s_{i-1} and normalized by the knot interval h_x as

$$r_x = (x - s_{i-1}) / h_x \,. \tag{3}$$

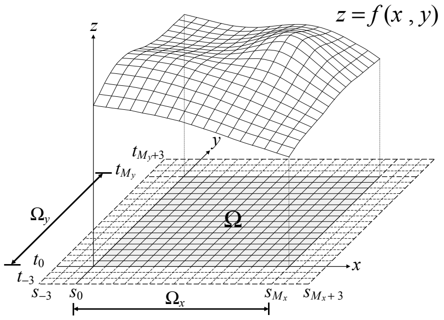


Figure 1. Equally-spaced knots s_i, t_j and surface $f(x, y)$ in Ω.

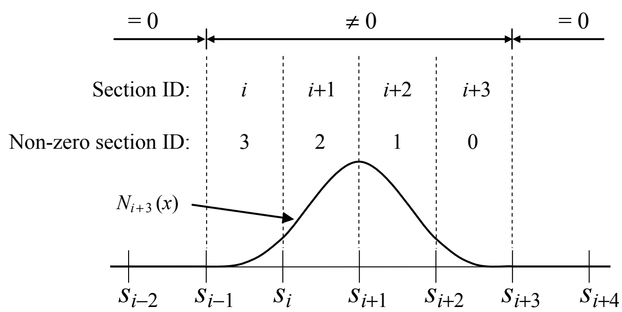


Figure 2. B-spline basis $N_{i+3}(x)$ that has non-zero value in $s_{i-1} \le x \le s_{i+3}$.

Provided that the ID number 3, 2, 1 and 0 are respectively assigned to four non-zero sections i, i + 1, i + 2 and i + 3 as shown in Figure 2, the bases in each non-zero section can be expressed by the following cubic polynomials;

$$B_1(r_x) = r_x^3 / 6 \tag{4a}$$

$$B_2(r_x) = (-3r_x^3 + 3r_x^2 + 3r_x + 1) / 6 \tag{4b}$$

$$B_3(r_x) = (3r_x^3 - 6r_x^2 + 4) / 6 \tag{4c}$$

$$B_4(r_x) = (-r_x^3 + 3r_x^2 - 3r_x + 1) / 6 \tag{4d}$$

based on de Boor-Cox algorithm (Ichida and Yoshimoto, 1979). For example, the basis in non-zero section 0 is expressed by the polynomial $B_4(r_x)$. As shown in Figure 3, the cubic polynomials given by $B_{4-i'}(r_x)$ (i' = 3, 2, 1, 0) clearly match the bases $N_i(x)$, $N_{i+1}(x)$, $N_{i+2}(x)$ and $N_{i+3}(x)$ respectively in section i ($s_{i-1} \le x \le s_i$). Thus, we can obtain the relation:

$$N_{i+i'}(x) = B_{4-i'}(r_x) \qquad (i' = 0, 1, 2, 3)\,. \tag{5}$$

2.3 Partial derivatives and definite integration of the surface

At a point (x, y) in Ω, there are at most four bases that have non-zero value with respect to both x and y. If a point (x, y) exists in a section (i, j), applying the relation (5) into equation (2), the surface $f(x, y)$ can be expressed by

$$f(x,y) = \sum_{i'=0}^{3}\sum_{j'=0}^{3} c_{i+i'\,j+j'} B_{4-i'}(r_x) B_{4-j'}(r_y)\,. \tag{6}$$

Thus, the first and second partial derivative of the surface $f(x, y)$ with respect to x and y can be expressed by

$$f_x(x,y) = \frac{1}{h_x}\sum_{i'=0}^{3}\sum_{j'=0}^{3} c_{i+i'\,j+j'} B'_{4-i'}(r_x) B_{4-j'}(r_y) \tag{7}$$

$$f_y(x,y) = \frac{1}{h_y}\sum_{i'=0}^{3}\sum_{j'=0}^{3} c_{i+i'\,j+j'} B_{4-i'}(r_x) B'_{4-j'}(r_y) \tag{8}$$

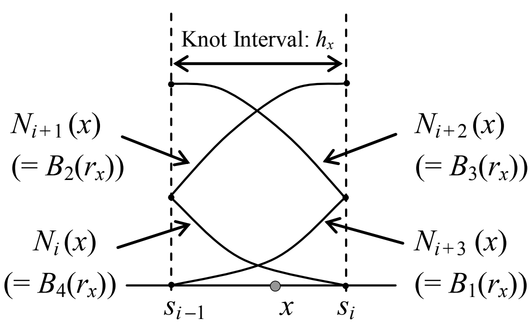


Figure 3. Non-zero B-spline bases $N_i(x)$, $N_{i+1}(x)$, $N_{i+2}(x)$ and $N_{i+3}(x)$ in $s_{i-1} \le x \le s_i$.

Table 1. Definite Integration values of cubic B-spline bases.

i	$\int_{\Omega_x} N_i(x)\mathrm{dx}$
$1, M_x + 3$	1 / 24
$2, M_x + 2$	1 / 2
$3, M_x + 1$	23 / 24
$[4, M_x]$	1

$$f_{xx}(x,y)=\frac{1}{h_x^2}\sum_{i'=0}^{3}\sum_{j'=0}^{3}c_{i+i'\,j+j'}B''_{4-i'}(r_x)B_{4-j'}(r_y) \tag{9}$$

$$f_{yy}(x,y)=\frac{1}{h_y^2}\sum_{i'=0}^{3}\sum_{j'=0}^{3}c_{i+i'\,j+j'}B_{4-i'}(r_x)B''_{4-j'}(r_y)\,. \tag{10}$$

On the other hands, in order to calculate the definite integration of the surface $f(x, y)$ with respect to Ω:

$$\iint_{\Omega} f(x,y)\,\mathrm{d}x=\sum_{i=1}^{M_x+3}\sum_{j=1}^{M_x+3}\left(c_{i\,j}\int_{\Omega_x}N_i(x)\,\mathrm{d}x\int_{\Omega_y}N_j(y)\,\mathrm{d}y\right), \tag{11}$$

we need the definite integrations of the bases $N_i(x)$ with respect to Ω_x:

$$\int_{\Omega_x}N_i(x)\,\mathrm{d}x \quad (i=1,2,\cdots,M_x+3)\,. \tag{12}$$

The equation (12) can be exactly calculated based on the equation (4a) to (4d) and relation (5). Table 1 shows the list of values derived from the equation (12). Consequently, the equation (11) can be calculated by using the values in Table 1.

3. TARGET GEOMORPHOLOGICAL CHARACTERISTICS

There are many kinds of geomorphological characteristics. In this study, we focused our attention on the following geomorphological characteristics that require approximate calculation in the case of DEM.

(1) Slope angle θ and slope trend ϕ: Slope angle is an angle between a normal vector of the surface and the z-axis, and slope trend is a direction of the slope measured clockwise from the y-axis. They are defined by the first partial derivatives of the surface;

$$\sin\theta=\frac{\sqrt{f_x(x,y)^2+f_y(x,y)^2}}{\sqrt{f_x(x,y)^2+f_y(x,y)^2+1}}\,,\ \cos\theta=\frac{1}{\sqrt{f_x(x,y)^2+f_y(x,y)^2+1}} \tag{13}$$

$$\sin\phi=\frac{-f_x(x,y)}{\sqrt{f_x(x,y)^2+f_y(x,y)^2}}\,,\ \cos\phi=\frac{-f_y(x,y)}{\sqrt{f_x(x,y)^2+f_y(x,y)^2}}\,. \tag{14}$$

(2) Laplacian L: Laplacian is the value to evaluate roughness of the surface. The greater this value, the rougher the surface is. It is defined by the second partial derivatives of the surface;

$$L=f_{xx}(x,y)+f_{yy}(x,y)\,. \tag{15}$$

(3) Volume of mountain body V_Ω: The volume of mountain body is generally defined by the volume upper than minimum height within Ω, but we simply defined it by the volume upper than x-y plane ($z = 0$);

$$V_\Omega = \iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y\,. \tag{16}$$

4. EXAMPLES OF EXTRANTION USING TEST DATA

A FORTRAN program was coded based on the algorithm. The program has the capability to calculate the geomorphological characteristics mentioned above as well as the value of the surface (height) and partial derivatives of the surface.

4.1 Verification of algorithm and program

In order to verify both algorithm and program, we tried to calculate the heights, first and second partial derivatives, and volume of mountain body from the surface estimated by BS-Horizon. In surface estimation, we used 361 × 361 height data generated by a function:

$$f(x,y) = 1 + \cos\frac{\pi}{180}x \sin\frac{\pi}{180}y \quad (x, y = 0, 1, \cdots, 360)\,. \tag{17}$$

Figure 4(a) shows the contour map for heights of the estimated surface at 1,801 × 1,801 points. Figure 4(b) and (c) show the profiles of the first and second partial derivatives with respect to y along A–A' (1,801 points) respectively. The estimated values almost perfectly accorded with the values from the equation (17). RMS errors between estimated values and theoretical values were (a) 3.8×10^{-6}, (b) 2.9×10^{-7} and (c) 2.0×10^{-7}. Furthermore, the estimated volume was 129,599.99 (theoretical volume is 129,600). Considering the figures of each value, we can say with fair certainty that both algorithm and program have no problems.

4.2 Application to real topography

As an example of application to real topography, we tried to calculate the heights, slope angles, slope trends and Laplacians from the surface generated from topographic map based on STRIPE method (Noumi, 2003) using BS-Horizon. Figure 5 shows the results. All figures were drawn using the values at 3,001 × 3,001 points. The results show the detailed changes of each value in whole domain. Especially in the case of slope angle and trend, the changing points of topography such as ridge lines and valley lines were detected as clear continuous lines as if they were drawn by human hands.

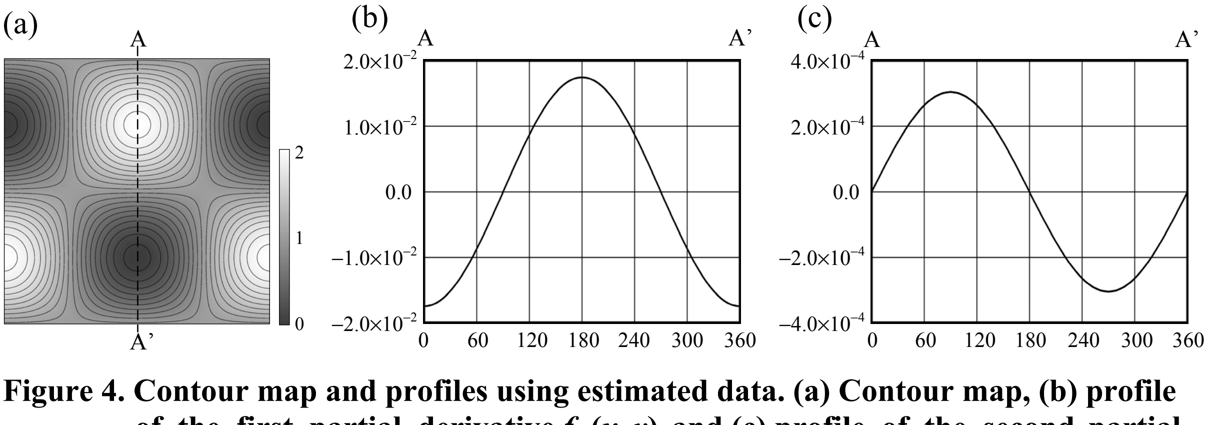
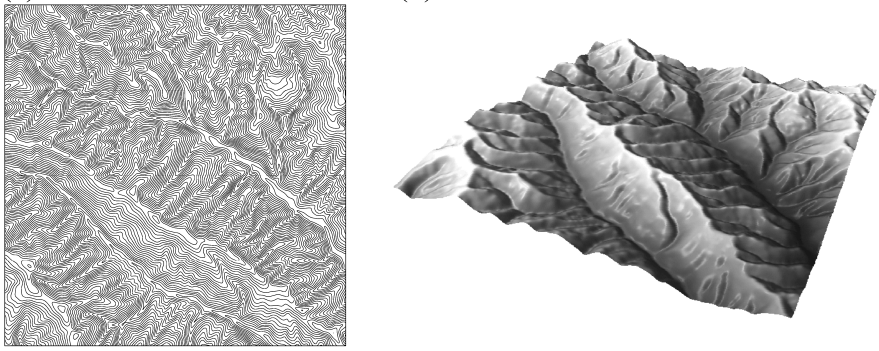


Figure 4. Contour map and profiles using estimated data. (a) Contour map, (b) profile of the first partial derivative $f_y(x, y)$ and (c) profile of the second partial derivative $f_{yy}(x, y)$.

(a) (b)



(c) (d) (e)

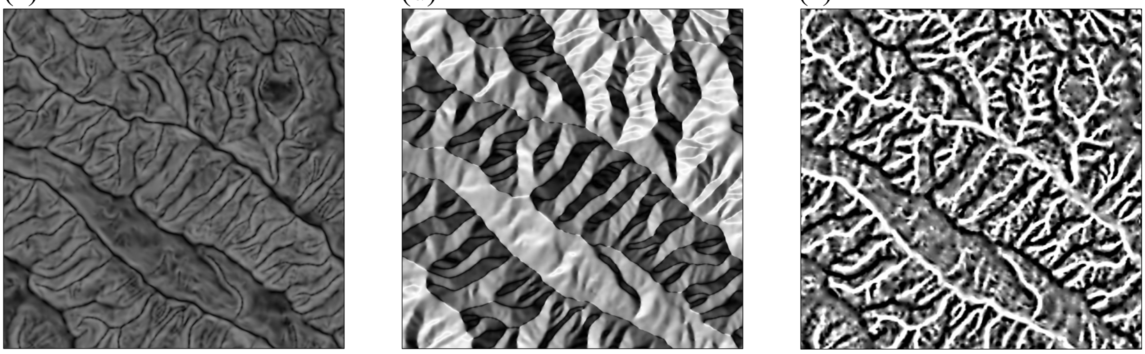


Figure 5. Results of extraction from real topography. (a) Contour map, (b) 3D visualization of the surface, (c) slope angle map, (d) slope trend map and (d) Laplacian map.

In the case of DEM, it is hard to estimate the partial derivatives exactly as Figure 4 and to get the precise result with continuity as Figure 5. These results clearly show that the analysis using the surface by BS-Horizon is much more useful than the one using DEM.

5. CONCLUSION

We presented a new algorithm for extraction of some major geomorphological characteristics based on the information of bi-cubic B-spline surface produced by BS-Horizon. Utilizing the surface by BS-Horizon, unlike DEM, we can do exact calculation including the partial derivatives and definite integration of the surface. We consider it leads to development of a new analysis technique, which is never realized by DEM, in the field of geomorphology and geology. A further direction of this study is to explore the utilization of the bi-cubic B-spline surface in the field of geology, especially in 3D geologic structure analysis.

6. REFERENCES

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